

On the mathematical origin of quantum space-time

G.SARDANASHVILY¹

Department of Theoretical Physics, Moscow State University, 117234 Moscow, Russia

Abstract An Euclidean topological space E is homeomorphic to the subset of δ -functions of the space $\mathcal{D}'(E)$ of Schwartz distributions on E . Herewith, any smooth function of compact support on E is extended onto $\mathcal{D}'(E)$. One can think of these extensions as sui generis quantum deformations. In quantum models, one therefore should replace integration of functions over E with that over $\mathcal{D}'(E)$.

A space-time in field theory, except noncommutative field theory, is traditionally described as a finite-dimensional smooth manifold, locally homeomorphic to an Euclidean topological space $E = \mathbb{R}^n$. The following fact (Proposition 1) enables us to think that a space-time might be a wider space of Schwartz distributions on E .

Let $E = \mathbb{R}^n$ be an Euclidean topological space. Let $\mathcal{D}(E)$ be the nuclear space of smooth complex functions of compact support on E . Its topological dual $\mathcal{D}'(E)$ is the space of Schwartz distributions on E , provided with the weak* topology [1, 2]. Since $\mathcal{D}(E)$ is reflexive and the strong topology on $\mathcal{D}'(E)$ is equivalent to the weak* one, $\mathcal{D}(E)$ is the a topological dual of $\mathcal{D}'(E)$. Therefore, any continuous form on $\mathcal{D}'(E)$ is completely determined by its restriction

$$\langle \phi, \delta_x \rangle = \int \phi(x') \delta(x - x') d^n x = \phi(x), \quad x \in E,$$

to the subset $T_\delta(E) \subset \mathcal{D}'(E)$ of δ -functions.

Proposition 1. *The assignment*

$$s_\delta : E \ni x \rightarrow \delta_x \in \mathcal{D}'(E) \tag{1}$$

is a homeomorphism of E onto the subset $T_\delta(E) \subset \mathcal{D}'(E)$ of δ -functions endowed with the relative topology (see Appendix for the proof).

As a consequence, $T_\delta(E)$ is isomorphic to the topological vector space E with respect to the operations $\delta_x \oplus \delta_{x'} = \delta_{x+x'}$, $\lambda \odot \delta_x = \delta_{\lambda x}$. Moreover, the injection $E \rightarrow T_\delta(E) \subset \mathcal{D}'(E)$ is smooth [3]. Therefore, we can identify E with a topological subspace $E = T_\delta(E)$ of the

¹E-mail: gennadi.sardanashvily@unicam.it

space of Schwartz distributions. Herewith, any smooth function ϕ of compact support on $E = T_\delta(E)$ is extended to a continuous form

$$\tilde{\phi}(w) = \langle \phi, w \rangle, \quad w \in \mathcal{D}'(E), \quad (2)$$

on the space of Schwartz distributions $\mathcal{D}'(E)$. One can think of this extension as being a quantum deformation of ϕ as follows.

The space $\mathcal{D}(E)$ is a dense subset of the Schwartz space $S(E)$ of smooth complex functions of rapid decrease on E . Moreover, the injection $\mathcal{D}(E) \rightarrow S(E)$ is continuous. The topological dual of $S(E)$ is the space $S'(E)$ of tempered distributions, which is a subset of the space $\mathcal{D}'(E)$ of Schwartz distributions. In QFT, one considers the Borchers algebra

$$A_S = \mathbb{C} \oplus S(E) \oplus S(E \oplus E) \oplus \cdots \oplus S(\bigoplus^k E) \oplus \cdots, \quad (3)$$

treated as a quantum algebra of scalar fields [4, 5]. Being provided with the inductive limit topology, the algebra A_S (3) is an involutive nuclear barreled LF-algebra [6]. It follows that a linear form f on A_S is continuous iff its restriction f_k to each $S(\bigoplus^k E)$ is well [1]. Therefore any continuous positive form on A_S is represented by a family of tempered distributions $W_k \in S'(\bigoplus^k E)$, $k = 1, \dots$, such that

$$f_k(\phi(x_1, \dots, x_k)) = \int W_k(x_1, \dots, x_k) \phi(x_1, \dots, x_k) d^n x_1 \cdots d^n x_k, \quad \phi \in S(\bigoplus^k E). \quad (4)$$

For instance, the states of scalar quantum fields on the Minkowski space \mathbb{R}^4 are described by the Wightman functions $W_k \in S'(\mathbb{R}^{4k})$ [2].

Any state of A_S is also a state of its subalgebra

$$A_{\mathcal{D}} = \mathbb{C} \oplus \mathcal{D}(E) \oplus \mathcal{D}(E \oplus E) \oplus \cdots \oplus \mathcal{D}(\bigoplus^k E) \oplus \cdots.$$

This quantization can be treated as follows. Given a function $\phi \in \mathcal{D}(\bigoplus^k E)$ on $\bigoplus^k E$, we have its quantum deformation

$$\hat{\phi} = \phi + f_k(\phi) \in C^\infty(\bigoplus^k E). \quad (5)$$

Let $\bigoplus^k E$ be identified to the subspace $T_\delta(\bigoplus^k E) \subset \mathcal{D}'(\bigoplus^k E)$ of δ -functions on $\bigoplus^k E$. Then the quantum deformation $\hat{\phi}$ (5) of ϕ comes from the extension of ϕ onto $\mathcal{D}'(\bigoplus^k E)$ by the formula

$$\hat{\phi}(z) = \phi(z + W_k), \quad z + W_k \in S'(\bigoplus^k E) \subset \mathcal{D}'(\bigoplus^k E).$$

Generalizing this construction, let us consider a continuous injection

$$s : \bigoplus^k E \ni z \rightarrow s_z \in \mathcal{D}'(\bigoplus^k E)$$

and a continuous function

$$s_\phi : \bigoplus^k E \ni z \rightarrow s_z(\phi) \in \mathbb{C}.$$

for any $\phi \in \mathcal{D}(\bigoplus^k E)$. For instance, the map s_δ (1) where $s_{\delta,\phi} = \phi$ is of this type. Given a function $\phi \in \mathcal{D}(\bigoplus^k E)$, we agree to call

$$\widehat{\phi} = \phi + s_\phi, \quad \widehat{\phi}(z) = \phi(z) + s_z(\phi) = \phi(z + s_z) \quad (6)$$

the quantum deformation of ϕ and to treat it as a function on the quantum space $\widehat{E} = (s_\delta + s)(E) \subset \mathcal{D}'(E)$.

For instance, let $\phi(x, y) \in \mathcal{D}(E \oplus E)$ be a symmetric function on $E \oplus E$. Then its quantum deformation (6) obeys the commutation relation

$$\widehat{\phi}(x, y) - \widehat{\phi}(y, x) = \langle \phi, s_{x,y} - s_{y,x} \rangle.$$

Let E be coordinated by (x^λ) , and let us consider a function $x^1 x^2$ on E , though it is not of compact support. Let us choose a map s such that all distributions s_x , $x \in E$, are of compact support. Its quantum deformation is $\widehat{x^1 x^2} = x^1 x^2 + s_x(x^1 x^2)$. It is readily observed that $\widehat{x^1 x^2} - \widehat{x^2 x^1} = 0$, i.e., coordinates on a quantum space commute with each other, in contrast to a space in noncommutative field theory.

Bearing in mind quantum deformations $\widehat{\phi}$ (2) of functions ϕ on E , one should replace integration of functions over E with that over $\mathcal{D}'(E)$. Here, we summarize the relevant material on integration over the space of Schwartz distributions $\mathcal{D}'(E)$.

I. Due to the homeomorphism (1), the space $T_\delta(E)$ is provided with the measure $d^n x$, invariant with respect to translations $\delta_x \rightarrow \delta_{x+a}$.

II. The space $M(E, \mathbb{C})$ of measures on E is the topological dual of the space $\mathcal{K}(E, \mathbb{C})$ of continuous functions of compact support on E endowed with the inductive limit topology (see Appendix). The space $M(E, \mathbb{C})$ is provided with the weak* topology. It is homeomorphic to a subspace of $\mathcal{D}'(E)$ provided with the relative topology. It follows that, for any measure ν on E , there exists an element $w_\nu \in \mathcal{D}'(E)$ and the Dirac measure ε_ν of support at w_ν such that, for each $\phi \in \mathcal{D}(E)$, we have

$$\int_E \phi \nu(x) = \langle \phi, w_\nu \rangle = \int_{\mathcal{D}'(E)} \langle \phi, w \rangle \varepsilon_\nu(w).$$

Let $T_x \subset \mathcal{D}'(E)$ denote a subspace of point measures $\lambda \delta_x$, $\lambda \in \mathbb{C}$, on $E = T_\delta(E)$. It is a Banach space with respect to the norm $\|\lambda \delta_x\| = |\lambda|$. Let us consider the direct product

$$T(E) = \prod_{x \in E} T_x. \quad (7)$$

By analogy with the notion of a Hilbert integral [7], we define the Banach space integral $(T(E), \mathcal{L}(E), d^n x)$ where $\mathcal{L}(E)$ is a set of fields

$$\hat{\varphi} : E \ni x \rightarrow \varphi_x \delta_x \in T(E)$$

such that:

- the range of $\mathcal{L}(E)$ is a vector subspace of the direct product $T(E)$ (7);
- there is a countable set $\{\varphi^i\}$ of elements of $\mathcal{L}(E)$ such that, for any $x \in E$, the set $\{\varphi_x^i\}$ is total in T_x ;
- the function $x \rightarrow \|\varphi_x\| = |\varphi_x|$ is $d^n x$ -integrable for any $\varphi \in \mathcal{L}(E)$.

Let $\mathcal{L}(E) = L^2(E, d^n x)$ be the space of complex square $d^n x$ -integrable functions on E . Clearly, $\mathcal{D}(E) \subset \mathcal{L}(E)$, and there is an injection $\mathcal{L}(E) \rightarrow M(E, \mathbb{C}) \subset \mathcal{D}'(E)$ such that

$$\varphi(\phi) = \int \phi(x') \varphi_x \delta(x' - x) d^n x d^n x'.$$

Therefore, let

$$\int \phi_x \delta_x d^n x$$

denote the image of φ in $\mathcal{D}'(E)$. Then any $d^n x$ -equivalent measure $\nu = c^2 d^n x$ (where $c \in L^2(E, d^n x)$ is strictly positive almost everywhere on E) defines the corresponding element

$$w_\nu = \int c^2(x) \delta_x d^n x$$

of $\mathcal{D}'(E)$. For instance, if $\nu = d^n x$, we have $\varphi_x = 1$ and

$$w_\nu = \int \delta_x d^n x.$$

III. Let Q be an arbitrary nuclear space (e.g., $\mathcal{D}(E)$, $S(E)$) and Q' its topological dual (e.g., $\mathcal{D}'(E)$, $S'(E)$). A complex function $Z(q)$ on Q is called positive-definite if $Z(0) = 1$ and

$$\sum_{i,j} Z(q_i - q_j) \bar{\lambda}_i \lambda_j \geq 0$$

for any finite set q_1, \dots, q_m of elements of Q and arbitrary complex numbers $\lambda_1, \dots, \lambda_m$. In accordance with the well-known Bochner theorem for nuclear spaces [8, 9, 10], any continuous positive-definite function $Z(q)$ on a nuclear space Q is the Fourier transform

$$Z(q) = \int \exp[i\langle q, w \rangle] \mu(w) \quad (8)$$

of a positive measure μ of total mass 1 on the dual Q' of Q , and *vice versa*.

Note that there is no translationally-invariant measure on Q' . Let a nuclear space Q be provided with a separately continuous non-degenerate Hermitian form $\langle \cdot | \cdot \rangle$. In the case of $Q = \mathcal{D}(E)$, we have

$$\langle \phi | \phi' \rangle = \int \phi \bar{\phi}' d^n x.$$

Let $w_q, q \in Q$, be an element of Q' given by the condition $\langle q', w_q \rangle = \langle q' | q \rangle$ for all $q' \in Q$. These elements form the image of the monomorphism $Q \rightarrow Q'$ determined by the Hermitian form $\langle \cdot | \cdot \rangle$ on Q . If a measure μ in (8) remains equivalent under translations

$$Q' \ni w \mapsto w + w_q \in Q', \quad \forall w_q \in Q \subset Q',$$

in Q' , it is called translationally quasi-invariant. However, it does not remain equivalent under an arbitrary translation in Q' , unless Q is finite-dimensional.

Gaussian measures exemplify translationally quasi-invariant measures on the dual Q' of a nuclear space Q . The Fourier transform of a Gaussian measure reads

$$Z(q) = \exp \left[-\frac{1}{2} B(q) \right],$$

where $B(q)$ is a seminorm on Q' called the covariance form. Let μ_K be a Gaussian measure on Q' whose Fourier transform

$$Z_K(q) = \exp \left[-\frac{1}{2} B_K(q) \right]$$

is characterized by the covariance form $B_K(q) = \langle K^{-1}q | K^{-1}q \rangle$, where K is a bounded invertible operator in the Hilbert completion \tilde{Q} of Q with respect to the Hermitian form $\langle \cdot | \cdot \rangle$. The Gaussian measure μ_K is translationally quasi-invariant. It is equivalent μ if

$$\text{Tr} \left(\mathbf{1} - \frac{1}{2} K K^+ \right) < \infty.$$

For instance, the Gaussian measures μ and μ' possessing the Fourier transforms

$$Z(q) = \exp[-\lambda^2 \langle q | q \rangle], \quad Z(q) = \exp[-\lambda'^2 \langle q | q \rangle] \quad \lambda, \lambda' \in \mathbb{R},$$

are not equivalent if $\lambda \neq \lambda'$.

If the function $\mathbb{R} \ni t \rightarrow Z(tq)$ is analytic on \mathbb{R} at $t = 0$ for all $q \in Q$, then one can show that the function $\langle q | u \rangle$ on Q' (e.g., the extension $\tilde{\phi}$ (2) of ϕ onto $\mathcal{D}'(E)$) is square μ -integrable for all $q \in Q$. Moreover, the correlation functions can be computed by the formula

$$\langle q_1 \cdots q_n \rangle = i^{-n} \frac{\partial}{\partial \alpha^1} \cdots \frac{\partial}{\partial \alpha^n} Z(\alpha^i q_i) |_{\alpha^i=0} = \int \langle q_1, w \rangle \cdots \langle q_n, w \rangle \mu(w).$$

In particular, an integral of the function $\tilde{\phi}$ (2) over $\mathcal{D}'(E)$ reads

$$\int \tilde{\phi} \mu(w) = \int \langle \phi' w \rangle \mu(w) = i \frac{\partial}{\partial \alpha} Z(\alpha \phi).$$

Appendix

Let $\mathcal{K}(E, \mathbb{C})$ be the space of continuous complex functions of compact support on $E = \mathbb{R}^n$. For each compact subset $K \subset E$, we have a seminorm

$$p_K(\phi) = \sup_{x \in K} |\phi(x)|$$

on $\mathcal{K}(E, \mathbb{C})$. These seminorms provide $\mathcal{K}(E, \mathbb{C})$ with the topology of compact convergence. At the same time, $\mathcal{K}(E, \mathbb{C})$ is a Banach space with respect to the norm

$$\|f\| = \sup_{x \in E} |\phi(x)|.$$

Its normed topology, called the topology of uniform convergence, is finer than the topology of compact convergence. The space $\mathcal{K}(E, \mathbb{C})$ can also be equipped with another topology, which is especially relevant to integration theory. For each compact subset $K \subset E$, let $\mathcal{K}_K(E, \mathbb{C})$ be the vector subspace of $\mathcal{K}(E, \mathbb{C})$ consisting of functions of support in K . Let \mathcal{U} be the set of all absolutely convex absorbent subsets U of $\mathcal{K}(E, \mathbb{C})$ such that, for every compact K , the set $U \cap \mathcal{K}_K(E, \mathbb{C})$ is a neighborhood of the origin in $\mathcal{K}_K(E, \mathbb{C})$ under the topology of uniform convergence on K . Then \mathcal{U} is a base of neighborhoods for the inductive limit topology on $\mathcal{K}(E, \mathbb{C})$ [11]. This is the finest topology such that the injection $\mathcal{K}_K(E, \mathbb{C}) \rightarrow \mathcal{K}(E, \mathbb{C})$ is continuous for each K . The inductive limit topology is finer than the topology of uniform convergence and, consequently, the topology of compact convergence. The space $M(E, \mathbb{C})$ of complex measures on E is the topological dual of $\mathcal{K}(E, \mathbb{C})$, endowed with the inductive limit topology. The space $M(E, \mathbb{C})$ is provided with the weak* topology, and $\mathcal{K}(E, \mathbb{C})$ is its topological dual. The following holds [10].

Lemma 2. *Let ε_x denote the Dirac measure of support at a point $x \in E$. The assignment*

$$s_\varepsilon : E \ni x \rightarrow \varepsilon_x \in M(E, \mathbb{C}) \tag{9}$$

is a homeomorphism of E onto the subset $T_\varepsilon \subset M(E, \mathbb{C})$ of Dirac measures endowed with the relative topology.

Of course, $\mathcal{D}(E) \subset \mathcal{K}(E, \mathbb{C})$, but the standard topology of $\mathcal{D}(E)$ is finer than its relative topology as a subset of $\mathcal{K}(E, \mathbb{C})$. Let $\mathcal{D}_R(E)$ denote $\mathcal{D}(E) \subset \mathcal{K}(E, \mathbb{C})$ provided with the relative topology, and let $\mathcal{D}'_R(E)$ be its topological dual endowed with the weak* topology.

Then $M(E, \mathbb{C})$ is homeomorphic to a subspace of $\mathcal{D}'_R(E)$ provided with the relative topology. At the same time, $\mathcal{D}'_R(E)$ is a subspace of $\mathcal{D}'(E)$ endowed with the relative topology. Thus, we have the morphisms

$$E \xrightarrow{s_\varepsilon} M(E, \mathbb{C}) \longrightarrow \mathcal{D}'_R(E) \longrightarrow \mathcal{D}'(E),$$

whose composition leads to the homeomorphism $x \rightarrow \varepsilon_x = \delta_x d^n x \rightarrow \delta_x (1)$.

References

- [1] F.Trevers, *Topological Vector Spaces, Distributions and Kernels* (Academic Press, New York, 1967).
- [2] N.Bogoliubov, A.Logunov, A.Oksak and I.Todorov, *General Principles of Quantum Field Theory* (Kluwer Acad. Publ., Dordrecht, 1990).
- [3] J.Colombeau, A multiplication of distributions, *J. Math. Anal. Appl.* **94** (1983) 96.
- [4] H.Borchers, Algebras of unbounded operators in quantum field theory, *Physica A* **124** (1984) 127.
- [5] S.Horuzhy, *Introduction to Algebraic Quantum Field Theory*, Mathematics and its Applications (Soviet Series) **19** (Kluwer Academic Publ. Group, Dordrecht, 1990).
- [6] A.Bélangier and G.Thomas, Positive forms on nuclear *-algebras and their integral representations, *Can. J. Math.* **42** (1990) 410.
- [7] J.Dixmier, *C*-Algebras* (North-Holland, Amsterdam, 1977).
- [8] S.Bochner, *Harmonic Analysis and the Theory of Probability* (California Univ. Press, Berkeley, 1960).
- [9] I.Gelfand and N.Vilenkin, *Generalized Functions, Vol.4* (Academic Press, New York, 1964).
- [10] N.Bourbaki, *Intégration* (Hermann, Paris, Chap. 1-4, 1965; Chap. 5, 1967; Chap. 9, 1969).
- [11] A.Robertson and W.Robertson, *Topological Vector Spaces* (Cambridge Univ. Press., Cambridge, 1973).